THE WRITING OF EXPLANATIONS AND JUSTIFICATIONS IN MATHEMATICS: DIFFERENCES AND DILEMMAS

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This paper reports on aspects of a study which investigated the writing of explanations and justifications in mathematics with Year 11 students. Six teachers and 36 students from the same school responded to problem solving tasks; 14 of these students were interviewed. Student and teacher views about the writing process are reported. The study suggests that students and teachers need to jointly negotiate an understanding of what is meant by an explanation, a justification, and what makes a quality response.

INTRODUCTION

In recent years there has been a growing interest in the importance of language factors in mathematics learning. Language is used in mathematics to make and clarify meaning. It provides a medium for creating, preserving, and communicating mathematical thinking (Brown, 1997). Communication is an integral feature of current curriculum reforms in mathematics. 'Mathematics in the New Zealand Curriculum' (Ministry of Education, 1992, p9) states that:

> The mathematics curriculum intended by this statement will provide opportunities for students to:
>
> develop the skills of presentation and critical appraisal of a mathematical argument or calculation, use mathematics to explore and conjecture, and learn from mistakes as well as successes;
>
> develop the characteristics of logical and systematic thinking, and apply these in mathematical and other contexts, including other subjects of the curriculum;
>
> develop the skills and confidence to use their own language, and the language of mathematics, to express mathematical ideas.

Communication in mathematics can be oral, written, formal, and informal. Writing in mathematics involves processes that are fundamental to learning and can be viewed as a reflective process, one which helps students clarify their thinking as they try to explain processes and demonstrate understandings in their own words (Clarke, Waywood, & Stephens, 1993; Pimm, 1987). Students' writings can be informative; providing access to how they think, illustrating misconceptions, patterns of thoughts and beliefs and indicating to the teacher students' variant conceptions of some notion (Burns, 1995; Pimm, 1987). Writing and speaking involve different levels or forms of cognitive activity (MacGregor, 1990). The focused articulation of one's thoughts is a higher order ability and one of the goals of mathematics education. Students may be used to operating at this higher cognitive level in verbal interactions; making judgements, justifying, and evaluating but not in written form, especially in mathematics (Ernest, 1989).

Writing varies according to the purpose and function of the text (genre). This study focused specifically on the writing genres of explanation and justification in mathematics. There are differences between these two writing genres: An *explanation* can be defined as making clear or telling why a state of affairs or an occurrence exists or happens whereas a *justification* provides grounds, evidence, or reasons to convince others (or to persuade ourselves) that a claim or assertion is true (Thomas, 1973). The explanation genre is seen to be predominantly the domain of the teacher. Only occasionally are opportunities given to students to perform similar linguistic demands (Marks & Mousley, 1990).

> Students should always be encouraged to think things through carefully, to understand, and to be able to explain. As students' arguments grow more sophisticated, the explanations should increasingly be conveyed in the formal language of mathematics (National Council of Teachers of Mathematics, 1998, p84).

Writing is viewed as a reflective process, one which helps students clarify their thinking as they try to explain processes and demonstrate understandings in their own words. It is from students' mathematical explanations, both oral and written that teachers endeavour to gain a perception of students' mathematical understandings. As one teacher wrote in a student's journal: "Unless you can explain it to me, you don't really understand" (Clarke, Waywood, & Stephens,1993, p249).

Justification is viewed as being most important to mathematicians.

> Students who cannot effectively explain the meaning of, and justify the use of, mathematical symbols, concepts, and operations are not yet fully fledged members of the community of discourse (Niemi, 1996, p361).

Mousley and Marks (1991) found in the classrooms they observed that students were asked to make mathematical judgements but rarely asked for an oral or written justification for their answers. They believe that much might be gained by asking students for expository answers to mathematical tasks.

In the mathematical community communication is in the form of discussion, argument, justification, and proof. The writing of explanations and justifications can lead students to the next fundamentally important step in mathematics; developing and evaluating mathematical arguments and proof. According to Silver, Kilpatrick, and Schlesinger (1990) much can be learned by providing justification and constructing a proof. Proving an assertion can lead to insights into further statements, to refinement of ideas, and modifications to improve the clarity and precision of justification.

The primary aim of this study was to find out how students respond, in writing, to questions requiring explanations and justifications, and secondly, to ascertain student and teacher views about the process. A detailed analysis of student responses to problems requiring written explanations and justifications is documented elsewhere (Bicknell, 1998). This paper reports on the student and teacher views about the writing of explanations and justifications in mathematics.

THE STUDY

The research was conducted at a large provincial co-educational secondary school with students coming from a full range of socio-economic backgrounds. There is an urban-rural mix and about 14% of the students are Maori. The school has a decile rating of 6[1]. The students who participated in the study came from the two highest ability grouped mathematics classes. The 14 interviewees were students aged 15 and 16 years in their third year at secondary school; seven came from each class. They were selected according to the following criteria: gender, and the type of responses given to the problem solving tasks.

Data collection instruments used in this study included a problem solving task sheet, interview, and questionnaire. The problem solving task sheet consisted of a set of word problems designed to generate students' written explanations and justifications. There were five problem solving tasks; one from each of the content strands of the curriculum: number, measurement, geometry, algebra, and statistics. The problems were based on those used in national School Certificate examinations, incorporated a range of prompts, and were designed to elicit a range of responses. An example of one problem is given in Figure 1.

Figure1
Example of a Problem

Derryn investigated the packaging of snack bars. She measured the packet with a ruler and found that it was 16.5 cm long, 9.2cm high, and 4.6 cm wide. She calculated the volume to be 698 cm^3 (3sf). The packet had 6 muesli bars in it. Each muesli bar was 7.5 cm long, 4 cm wide but the thickness of the bars varied between 2.6cm and 2.8cm.

Find the volume of the 6 muesli bars as a percentage of the volume of the packet.

Explain what you are calculating at each step and show your working.

Round your answer appropriately, stating the degree of accuracy.

Justify why you have chosen this degree of accuracy.

Semi-structured interviews, conducted with 14 of the students, were used to obtain information about students' attitudes, perceptions, and beliefs, and also to question students about responses to the problems posed in the problem solving task sheet. A questionnaire was completed by six teachers in the mathematics department in order to gain an understanding of the teachers' views and perceptions of the writing of explanations and justifications in mathematics.

RESULTS AND DISCUSSION

Teacher Views

The teachers all agreed that the writing of explanations and justifications is an important aspect of the mathematical processes and recognized advantages to students' learning in mathematics. It was articulated that the writing process helped students to sort out and clarify understanding of concepts in their own minds and thereby understand the work that they were doing. The teachers believed that the process of writing explanations and justifications helped students "develop their reasoning and analytical ability" and their "communication skills". More specifically, it was viewed as helping students "crystallize the task" and a way to "encourage alternative ideas and methods". Teachers felt that it will "often show up other areas for developing and exploring while they are working through it". Having students write justifications can reduce the "guess and leave it at that" approach. Written explanations were seen as particularly useful for helping diagnose students' difficulties or misconceptions.

The teachers all believed that the writing of explanations and justifications should be specifically taught. The teacher's role was seen as important in helping students develop skills in efficient and systematic recording methods, to help improve the clarity and specificity of explanations. To assist students in developing these skills the teachers felt that regular practice should be provided, especially through investigations and assignments. The teachers all acknowledged giving their students opportunities in class to develop skills in the writing of explanations and justifications. Students are regularly encouraged to explain, although for most of the teachers this is in oral situations.

> I insist that they should explain what they did to get the answer. (Teacher A)
>
> I get students to explain at every opportunity - it helps students to clarify their understanding of a concept. (Teacher C)

One teacher uses a strategy of having students write out their explanations first and then picks volunteers to read out their answers to the rest of the class. Another of the teachers uses group work as an opportunity to encourage students to provide oral justifications.

The teachers all expressed the view that students acquire an understanding of what constitutes a good explanation from being given opportunities to write explanations to problems posed. They felt that this process should begin to be addressed in the primary school. In order to encourage students' awareness and development of quality responses, the teachers felt that students should be encouraged to attempt explanations, share responses, and consider the range of answers to a problem. It was considered important that students be given examples of good explanations so that they can evaluate them and make comparisons with their own responses. Modelling by the teacher, both orally and in writing, was also considered a useful strategy for helping students develop an understanding and awareness of what makes a quality response.

Teacher Concerns

Despite both the curriculum and external assessments emphasising the importance of students being able to communicate mathematical findings teachers were concerned about whether this is happening in practice.

> It's in the too hard basket and is promised but shelved. A sudden burst of 'have to do Mathematical Processes' and so questions are put into the test. However, it's good to see thinking not rote learning more in balance. (Teacher D)

There was a common concern about the time factor involved when incorporating this approach to writing in the mathematics class. While time is a problem, Teacher C contends that "the value in the learning process far outweighs this disadvantage".

> The curriculum needs changing with some content eliminated - this sort of work requires hands-on, investigative work which takes time. As maths teachers some decisions need to be made as to what is worthwhile and then worked on with greater time for reflective thinking, writing, and analysing. (Teacher F)

The teachers expressed a common concern about the lack of support given to them to develop expertise in the teaching of the mathematical processes. The limited support that they had received had come from both internal and external sources. Internally, some support had come from departmental meetings and discussions. This was viewed as one way of providing collegial support in learning how to teach the writing of explanations and justifications. However, as one teacher commented: "Basically there's been trial and lots of error. The students themselves have probably had the most input into my work." The teachers had also addressed the issue by trying to find exemplars. Concerns were also expressed that there are distinct disadvantages for students weak in language skills.

Student Views

Students supported the teachers in believing that writing explanations is an important process in mathematics. In examining students' comments it became apparent that they believed that the writing of explanations and justifications helped develop their mathematical understanding and metacognition.

> It's a little unexpected having to write it down but it is important because it personally helps me understand it more. I know what I'm doing if I write it down. It's easier to look back and say: 'I know why I put that number there' . (Paula)
>
> It makes you have to think about what you're actually doing. That, sometimes helps you understand and puts you back on track. (Anne)

This viewpoint was supported by other students who also acknowledged that it helps them identify possible errors and track down specifically where they went wrong in the process of explaining an answer.

> If you get things wrong you can go back through and catch on to where you've gone wrong. It helps you to go through the steps. If you just write down an answer you don't know where you've gone wrong. It's better than just writing down an answer. (Kim)
>
> You know that you understand it if you're able to write out an explanation. (Jessica)

Many students acknowledged that writing out an explanation also helped them remember what they had learned.

> It's revision, too, it jogs my memory about how to do it rather than just writing down the answer. (Paula)

Some of the students clearly identified benefits, outside of the mathematics classroom, from being able to write explanations and justifications. Interestingly, the teachers had not articulated this link to the 'real world'.

> It's pretty important as it shows that you understand the question and in the workforce they're wouldn't give it to you as a question just with the numbers. They'd want you to show some understanding and put all the pieces together. (Mark)
>
> If you are putting it to work in daily life you can't just write out answers. (Jessica)
>
> You need to be able to write explanations and justifications for jobs and work these days, they don't just want the straight answer. They try to find out if you know what you've got to do with the answer. It makes you think about it more. (John)

Student Concerns

Despite most of the students identifying positive benefits of the writing process, two of the students expressed a dislike for writing explanations and justifications in mathematics. Being expected to write explanations and justifications was perceived as a nuisance, something that helped fill in examination time, and really was quite a tedious exercise.

> It helps you a bit in the beginning when you're becoming familiar with something but after you've been doing it for a while it becomes boring and in the exams you spend too much time writing. (Anthony)
>
> I find it a real pain. We know what we're doing and you can look and see where we got the numbers from or you can see it on a diagram, but writing it down, 'sux!' (Charles)

Charles finds writing explanations a frustrating process. First, he carries out the numerical calculation and then he returns to write out the explanation. He explains;

> I find it a pain doing the question twice. You work it all out and then you have to read through and think 'what did I do there and what do I do here?' You have to write out all that you did, yet what you did is all in the answer. It's just putting it into words which is a pain! I find it heaps easier just doing the maths; I get stuck sometimes writing it out. You end up writing half a page. I find it easier just doing the numerical side.

Some students found writing explanations difficult, not essentially because of the writing process, but the need to spend time on reading and comprehending the problem.

> I don't like some problems which are complicated ones with heaps of writing and problems where they put in heaps of other stuff. It's really confusing, where you have to sort out what they actually want you to do, what formula they want you to use. (Jan)
>
> I don't like big long word problems. I don't like the proof reading, reading, and trying to work everything out. (Fay)
>
> You read the question and then you end up having to read them over and over again. Then you try and start working things out. You get worried about time, you're just sitting there and everyone else is writing down pages. (Anne)

For some students the real-life context actually got in the way of the problem solving process causing concern and confusion. The problems in this case did not contain

superfluous or insufficient information but one problem, for example, required students to understand the procedure of 'trading in', another used the building of a shed as a way of 'dressing up' trigonometry in a realistic situation. The context of some problems isolated students from being able to interpret the problem as intended by the writer.

> I get put off by heaps of writing and problem solving especially if you put in real-life. If you put in a diagram it's easier. I hate having to sort everything out. (Merryn)

Not only the context but the language used in the problems also appeared to effect students' success in solving the problems:

> A lot of subjects and setting gets in the way. The way that they word things is stupid. They just word it really difficult to make it harder for you to understand. That's the worst part of it. (Paula)

Several students commented on the 'wordiness' of problems and having to spend time reading, interpreting, and often re-reading a problem posed. Time consequently became an issue for the students who could not read and comprehend problems as quickly as others.

In mathematics it is not how much or how neatly students write but what they write that is important. However, students expressed concerns about not knowing how much they should write. For the problems presented in this study, no lines were provided for the students to write on but a reasonable space was allocated. For some students the amount of space influenced how much they thought it was intended that they should write.

> I sometimes think about the space and if I do just the calculations I know it won't take up the space. That's why I write but then it goes over. (Jessica)[2]

> You feel like you need to fill up the whole space. The space provided makes you think that you have to write heaps. (Paula)

The common misconception was that students are expected to 'fill' the allocated space when a 'write-on' script is provided.

Despite the use of comprehensive prompts such as 'explain what you are calculating at each step' and 'clearly explain your working so that someone else can understand it' students still expressed concern about not knowing how to write an explanation and what was expected. Students used varying modes of representations for an explanation from purely symbolic to a combination of symbols, diagrams, and words. Regardless of whether they wrote detailed answers or only recorded calculations, many of these students expressed doubt about whether they had responded to the problems as intended.

The students experienced difficulties writing justifications. They did not understand what it meant to justify an answer and the majority of the students in the study avoided writing justifications as prompted by the questions.

> I really don't know what they mean. Justify is not really my sort of word, like, it's a **big** word. Too big, usually you're asked to explain, not justify. (Kim)

Richard makes a comparison between the prompts 'explain' and 'justify', and articulates how he feels when he meets these two instructions in a problem:

> It's quite hard to understand what it is that they want you to write. You don't meet the word justify that often.... Explain and justify, they're sort of the same thing. If it was explain I could probably do it a lot easier. When you come across justify you think that they want a really long answer or why you have chosen this. If it was explain you could do it in a sentence or so, but justify you have to think more about what it is they want you to write down.

Common Concern

A key concern for the students was not knowing what makes a quality response. This issue was also raised by some of the teachers who recognized the need to develop an understanding

of what is meant by a quality response. Students realize that their intended audience is usually the teacher or examiner and are therefore usually motivated to do their best. However, it is knowing what students should be aiming to achieve that is problematic when there are few suitable exemplars available to students and teachers. A report of brief comments from an examiner's report is insufficient feedback to develop teacher knowledge and expertise of what makes a quality response. As schools incorporate more internal assessment and greater use of open-response items, informed reliable professional judgement becomes paramount.

CONCLUSION

Tasks requiring students to write explanations and justifications are consistent with the aims of the curriculum focusing on communication in mathematics. The teachers recognized the importance of writing explanations and justifications and believe that the process should be specifically taught. They felt that the writing process helps students develop skills in thinking and communicating. Examining students' written explanations and justifications gives teachers an opportunity to diagnose students' difficulties or misconceptions. The teachers suggested that they could help students by modelling good examples, encouraging class discussion, and by involving students in the evaluation process. They viewed it as important that both students and teachers acquire an understanding of what makes a quality response.

The teachers identified a number of barriers when endeavouring to incorporate this aspect of communication into their classes. Concern was also expressed about the time factor and the lack of teacher resource material. They expressed concerns that the writing process could cause difficulties for some students who are not strong linguistically and it could therefore be a threatening process.

The majority of students felt positive about the writing of explanations and believed that it helped develop their thinking skills. They found that it helped them remember and confirm new mathematical understandings. Students need to be presented with a range of mathematical problems to be solved and encouraged to write explanations and justifications as part of the problem solving process. Concerns were expressed about the reading and comprehending of problems, knowing how much to write, and interpreting the context. The students also were not confident about knowing what makes a quality explanation and did not know what it meant to justify.

A learning environment should be established in which writing is a regular and natural part of classroom discourse so that students become more confident at writing explanations and justifications. This study has shown that students believe that writing explanations is an important process in mathematics. However, the students viewed the writing of explanations differently from the writing of justifications. They showed a reluctance to write justifications and had a poor understanding of what it means to justify. Together, both students and teachers face the dilemma of not really knowing what makes a quality response.

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Endnotes

[1] Each state and integrated school is ranked into deciles, low to high, on the basis of an indicator. The indicator used measures the extent to which schools draw from low socio-economic communities.
[2] Jessica writes prolifically in all her responses and frequently uses the blank back of a page to continue her response.